XXVIII. *On the Summation of Series, whoſe general Term is a determinate Function of* z *the Diſtance from the firſt Term of the Series. By* Edward Waring, *M. D.* Lucaſian *Profeſſor of the Mathematics at* Cambridge, *and Fellow of the Societies of* London *and* Bononia.

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PROBLEM.

THE ſum S *being given, to find a ſeries of which it is the ſum.*

1. Reduce the ſum S into a converging ſeries, proceeding according to the dimenſions of any ſmall quantities, and it is done. For example: let any algebraical function S of an unknown or ſmall quantity x be aſſumed, reduce it into a converging ſeries proceeding according to the dimenſions of x, and there reſults a ſeries whoſe ſum is S. 2. Let A, B, C, &c. be algebraical functions of x; reduce the $\int A\dot{x}$, $\int B\dot{x}$, $\int C\dot{x}$, &c. into a converging ſeries, proceeding according to the dimenſions of x, and the problem is done.

It is always neceſſary to find the values of x, between which the abovementioned ſerieſes converge. Reduce the algebraical function S in the firſt example, and the algebraical functions A, B, C, &c. in the ſecond into their loweſt terms; and in ſuch a manner, that the quantities contained in the numerator and denominator may have no denominator: make the denominator in the firſt example, and the denominator in the ſecond, and every diſtinct irrational quantity contained in them reſpectively $=0$; and alſo every diſtinct irrational quantity contained in the numerators $=0$. Suppoſe α the leaſt root affirmative or negative (but not $=0$) of the abovementioned reſulting equations; then a ſeries aſcending according to the dimenſions of x will always converge, if the value of x is contained between α and $-\alpha$; but if x be greater than α or $-\alpha$, the abovementioned ſeries will diverge. Let π be the greateſt root of the abovementioned reſulting equations; then a ſeries deſcending according to the reciprocal dimenſions of x will converge, if x be greater than $\pm\pi$; but, if leſs, not. When impoſſible roots $a\pm b\sqrt{-1}$ are contained in the equations, an aſcending ſeries will converge, if x be leſs than the leaſt root $\pm\alpha$, and $\pm(a-b)$ and $\pm(a+b)$; or more generally, if x be leſs than the leaſt root $\pm\alpha$, and x^{n+1} at an infinite diſtance n, be infinitely leſs than

$$\frac{2a^n - 2 \, . \, n \, . \, \frac{n-1}{2} a^{n-2} b^2 + 2 \, . \, n \, . \, \frac{n-1}{2} \, . \, \frac{n-2}{3} \, . \, \frac{n-3}{4} a^{n-4} b^4 - \&c.}{(a^2+b^2)^n}:$$

a deſcending ſeries will always converge, when x is greater than the greateſt root of the reſulting equations; and x^{n-1}, when n is infinite, is infinitely greater than $(a+b)^n$ and $(a-b)^n$; or more generally than $2a^n - 2n \, . \, \frac{n-1}{2} a^{n-2} b^2 + 2n \, . \, \frac{n-1}{2} \, . \, \frac{n-2}{3} \, . \, \frac{n-3}{4} a^{n-4} b^4 - \&c.$

This follows from Caput 3. of the Meditationes Algebraicæ.

Cor. It appears from hence, that, if the aſcending ſeries converges, the deſcending will diverge; and, *vice verſâ*, if the deſcending converges, the aſcending will diverge, unleſs all the roots of the above-mentioned reſulting equations may be deemed of equal magnitude, as $+\alpha$ and α, $\alpha\sqrt[n]{-1}$, &c. and $x=\alpha$; in which caſe ſometimes both ſerieſes may become the ſame converging ſeries, &c.

When x, in the preceding caſes, is equal to the leaſt or greateſt root, the ſeries will ſometimes converge, and ſometimes not, as is ſhewn in the above-mentioned chapter. Whether the ſum of a ſeries, whoſe general term is given, can be found or not, will in many caſes appear from the law of the multinomial and other more general ſerieſes.

2. There are ſerieſes which always converge, whatever may be the value of x; as, for example, the ſeries $1+\frac{1}{2\cdot 3}x+\frac{1}{2\cdot 3\cdot 4\cdot 5}x^2+$&c. or $1+\frac{x}{2^2}+\frac{x^2}{3^3}+\frac{x^3}{4^4}+$&c. &c. always converge, whatever may be the value of x; but it may be obſerved, that theſe ſerieſes never ariſe from the expanſion of algebraical functions of x, or the before-mentioned fluents; but, in a few caſes, they may from fluxional equations. There are alſo ſerieſes which never converge as $1+1\cdot 2x+1\cdot 2\cdot 3x^2+1\cdot 2\cdot 3\cdot 4x^3+$&c. to which the preceding remark may be applied.

3. In the year 1757 ſome papers, which contained the firſt edition of my Meditationes Algebraicæ, were ſent to the Royal Society, in which was contained the following rule, *viz.* let S be a given function of the quantity x, which expand into a ſeries ($a+bx^m+cx^{2m}+$&c.) proceeding according to the dimenſions of x; in the quantity S, for x^m write αx^m, βx^m, γx^m, &c. where α, β, γ, &c. are roots of the equation $x^n-1=0$; and let the reſulting quantities be A, B, C, D, &c. then will $\frac{A+B+C+D+\&c.}{n}$ be equal to the ſum of the firſt, $2n+1$, $3n+1$, &c. terms *in infinitum.* This method, in the preface to the

laſt edition of the Meditationes Algebraicæ, is rendered more correct and general.

4. Let the ſum of a ſeries required be expreſſed by a function of a quantity z, the diſtance from the firſt term of the ſeries, then will the general term be the difference between the two ſucceſſive ſums generally expreſſed.

5. Let the general term be an algebraical function of z: 1ſt, let it be $\frac{az^m + bz^{m-1} + cz^{m-2} + \&c.}{z+e \cdot z+e+1 \cdot z+e+2 \ldots z+e+n-1} = \mathrm{T}$, where m and n are whole numbers; and m (if the ſum of an infinite ſeries of terms is required) leſs than n by two or more: then the general term $\frac{az^m + bz^{m-1} + \&c.}{z+e \cdot z+e+1 \cdot z+e+2 \ldots z+e+n-1} =$

$$\frac{\gamma}{z+e \cdot z+e+1} + \frac{\delta}{z+e \cdot z+e+1 \cdot z+e+2} +$$

$$\frac{\varepsilon}{z+e \cdot z+e+1 \cdot z+e+2 \cdot z+e+3} + \&c. \ldots \frac{\theta}{z+e \cdot z+e+1 \ldots z+e+n-1},$$

whence if

$z+e+2 \cdot z+e+3 \cdot z+e+4 \ldots z+e+n-1 = z^{n-2} + \mathrm{A}z^{n-3} + \mathrm{B}z^{n-4} + \&c.$;

$z+e+3 \cdot z+e+4 \cdot z+e+5 \ldots z+e+n-1 = z^{n-3} + \mathrm{A}'z^{n-4} + \mathrm{B}'z^{n-5} + \&c.$;

$z+e+4 \cdot z+e+5 \ldots z+e+n-1 = z^{n-4} + \mathrm{A}''z^{n-5} + \mathrm{B}''z^{n-6} + \&c.$ and ſo on; then, if $m = n-2$, will $\gamma = a$, $\delta = b - \gamma\mathrm{A}$, $\varepsilon = c - \delta\mathrm{A}' - \gamma\mathrm{B}$, $\zeta = d - \varepsilon\mathrm{A}'' - \delta\mathrm{B}' - \gamma\mathrm{C}$, &c.; whence the integral *in infinitum*, or ſum of the infinite ſeries, will be $\frac{\gamma}{z+e} + \frac{\delta}{2 \cdot z+e \cdot z+e+1} +$

$$\frac{\varepsilon}{3 \cdot z+e \cdot z+e+1 \cdot z+e+2} + \&c.$$

The reduction of the general term T into quantities of the before given formulæ was publiſhed in the Meditationes, printed in the year 1774. It was before reduced into formulæ of the ſame kind nearly by Mr. Nichole in the Paris Acts.

2d, Let the general term be $\mathrm{T}' =$

$$\frac{az^b + bz^{b-1} + cz^{b-2} + \&c.}{z+e \cdot z+e+1 \cdot z+e+2 \ldots z+e+n-1 \times z+f \cdot z+f+1 \ldots z+f+m-1 \times z+g \cdot z+g+1 \ldots z+g+l-1 \times \&c}$$

where b is a whole number leſs than $n+m+l+\&c.$ (if it be greater, then the fraction can eaſily be reduced into a rational quantity $az^{b-n-m-l-\&c.}+\&c.$ and a fraction of the before-mentioned kind); then will $T' = \left(\frac{\alpha}{z+e} + \frac{\alpha'}{z+f} + \frac{\alpha''}{z+g} + \&c.\right) +$

$$\left(\frac{\beta}{z+e\,.\,z+e+1} + \frac{\beta'}{z+f\,.\,z+f+1} + \frac{\beta''}{z+g\,.\,z+g+1} + \&c.\right) +$$

$$\left(\frac{\gamma}{z+e\,.\,z+e+1\,.\,z+e+2} + \frac{\gamma'}{z+f\,.\,z+f+1\,.\,z+f+2} + \frac{\gamma''}{z+g\,.\,z+g+1\,.\,z+g+2}\right.$$

$$\left.+\&c.\right) \ldots \left(\frac{X}{z+e\,.\,z+e+1\,..\,z+e+n-1} + \frac{X'}{z+f\,.\,z+f+1\,..\,z+f+m-1}\right.$$

$\left. + \frac{X''}{z+g\,.\,z+g+1\,..\,z+g+l-1} + \&c.\right)$; whence its integral *in infinitum*, that is, the ſum of the infinite ſeries can be found when $\alpha=0$, $\alpha'=0$, $\alpha''=0$, &c.; and conſequently b not greater than $n+m+l+\&c.-2$; otherwiſe not. If b is not greater than $n+m+l+\&c.-2$, then will $\alpha+\alpha'+\alpha''+\&c.=0$, for elſe the ſum would be infinite.

Let the number of quantities (e, f, g, &c.) be r, then from r independent integrals of a ſeries, whoſe term is T'; or from $(r-1)$ independent ſums of infinite ſerieſes, whoſe term is T'; that is, where b is not greater than $n+m+l+\&c.-2$; can be deduced the ſum of all infinite ſerieſes of the before-mentioned formulæ, whoſe general term is T'.

If any factors are deficient in the denominator, as ſuppoſe the term to be $z+e \times z+e+3 \times z+e+n-1$; multiply the numerator and denominator by the deficient factors, *viz.* by $z+e+1\,.\,z+e+2 \times z+e+4\,.\,z+e+5\,..\,z+e+n-2$, and it acquires the preceding formula; and ſo in the following examples.

3d, Let the denominator be $\overline{x+e}^{\pi} \times \overline{x+e+1}^{\pi} \times \overline{x+e+2}^{\pi} \ldots \overline{x+e+n-1}^{\pi} \times \overline{x+e+\mu}^{\pi'} \times \overline{x+e+\mu+1}^{\pi'} \times$

$x+e$

$\overline{x+e+\mu+2}^{\,\pi'} \times$ &c. $\times \overline{x+f}^{\,\varrho} \times \overline{x+f+1}^{\,\varrho} \times \overline{x+f+2}^{\,\varrho} \ldots \times \overline{x+f+m-1}^{\,\varrho} \times$ &c. $=\mathrm{D}$, where π, π', ϱ, &c.; μ, &c. are whole numbers; and the general term is $\frac{az^{b}+bz^{b-1}+cz^{b-2}+\&c.}{\mathrm{D}}$ $=\mathrm{T}''$; then, if the dimenſions of z in the numerator be leſs than its dimenſions in the denominator, will $\mathrm{T}''=$

$$\left(\frac{\alpha}{z+e}+\frac{\alpha'}{(z+e)^{2}}+\frac{\alpha''}{(z+e)^{3}}\ldots\frac{\alpha'^{\pi+\pi'-1}}{(z+e)^{\pi+\pi'}}+\frac{\beta}{z+f}+\frac{\beta'}{(z+f)^{2}}+\frac{\beta''}{(z+f)^{3}}\ldots\frac{\beta'^{\varrho-1}}{(z+f)^{\varrho}}+\&c.\right)+\left(\frac{\gamma}{z+e\,.\,z+e+1}+\frac{\theta}{z+f\,.\,z+f+1}+\&c.\right)+\&c.;$$

and in general there will be included all terms of the formulæ,

$$\frac{\mathrm{A}\left(\overline{z+e+i}^{\,p}-\overline{z+e}^{\,p}\right)}{(z+e)^{p}\,.\,(z+e+1)^{p}\ldots(z+e+i)^{p}},$$

$$\frac{\mathrm{B}\left((z+f+i')^{p'}-(z+f)^{p'}\right)}{(z+f)^{p'}\,.\,(z+f+1)^{p'}\ldots(z+f+i')^{p'}},\ \&c.$$

$$\frac{\mathrm{C}\left((z+e+\mu+i'')^{p''}-(z+e+\mu)^{p''}\right)}{(z+e+\mu)^{p''}\,.\,(z+e+\mu+1)^{p''}\ldots(z+e+\mu+i'')^{p''}},\ \&c.$$

where A, B, C, &c. α, α', &c. β, β', &c. γ, θ, &c. denote invariable quantities; and p, p', p'', &c. are whole numbers not greater than π, ϱ, π', &c. reſpectively; and i, i', i'', &c. are whole numbers not greater than $n-1$, $m-1$, &c.

If all the quantities α, α', α'', &c. β, β', β'', &c. &c. are $=0$, the ſum of the ſeries can be expreſſed in finite terms of the quantity z, otherwiſe not; and alſo if b be leſs than the dimenſions of z in the denominator by two or more, then will $\alpha+\beta+\&c.=0$, otherwiſe the ſum would be infinite.

From $\pi+\pi'+\varrho+\&c.-1$ independent ſums of infinite ſerieſes of this kind can be deduced the ſums of all infinite ſerieſes of the ſame kind.

This method may be extended to infinite ſeries, in which exponentials as e^z are contained, which will eaſily be ſeen from ſome ſubſequent propoſitions; but in my opinion the ſubſequent method of finding the ſum of ſerieſes is to be preferred to the preceding one, both for its generality and facility.

6. 1. Let the general term be $(az^h + bz^{h-1} + cz^{h-2} + \&c.) \times (z+e)^{-1} . (z+e+1)^{-1} . (z+e+2)^{-1} \ldots (z+e+n-1)^{-1}$; where h is a whole number leſs than n by two or more, when the ſum of an infinite ſeries is required.

Aſſume for the ſum the quantity $(z+e)^{-1} . (z+e+1)^{-1} . (z+e+2)^{-1} \ldots (z+e+n-2)^{-1} \times (\alpha z^{h'} + \beta z^{h'-1} + \gamma z^{h'-2} + \&c.)$; find the difference between this ſum and its ſucceſſive one $(z+e+1)^{-1} . (z+e+2)^{-1} . (z+e+3)^{-1} \ldots (z+e+n-1)^{-1} \times (\alpha \overline{z+1}^{h'} + \beta \overline{z+1}^{h'-1} + \&c.)$, which will be $-(z+e)^{-1} . (z+e+1)^{-1} . (z+e+2)^{-1} \ldots (z+e+n-1)^{-1} \times (\overline{z+e} \times (\alpha \overline{z+1}^{h'} + \beta \overline{z+1}^{h'-1} + \&c.) - \overline{z+e+n-1} \times (\alpha z^{h'} + \beta z^{h'-1} + \&c.) = \overline{h'-n+1}\, \alpha z^{h'} + \&c.)$; then make the terms of this difference equal to the correſpondent terms of the given quantity $az^h + bz^{h-1} + \&c.$ and there reſult $h' = h$, $-\overline{h-n+1} \times \alpha = a$, and conſequently $\alpha = \frac{-a}{h-n+1}$, &c.

2. Let the general term be $(z+e)^{-1} . (z+e+1)^{-1} . (z+e+2)^{-1} \ldots (z+e+n-1)^{-1} \times (z+f)^{-1} . (z+f+1)^{-1} . (z+f+2)^{-1} \ldots (z+f+m-1)^{-1} \times (az^h + bz^{h-1} + cz^{h-2} + \&c.)$. Aſſume the quantity $(z+e)^{-1} . (z+e+1)^{-1} \ldots (z+e+n-2)^{-1} \times (z+f)^{-1} . (z+f+1)^{-1} . (z+f+2)^{-1} \ldots (z+f+m-2)^{-1} \times (\alpha z^{h'} + \beta z^{h'-1} + \gamma z^{h'-2} + \&c.)$ for the ſum of the ſeries ſought; and thence deduce the general term, which ſuppoſe equal to the given general term, and from equating their correſponding parts eaſily can be deduced the index h' and co-efficients α, β, γ, &c. and conſequently the ſum of the ſeries ſought.

3. Let the general term reduced to its loweſt dimenſions be $\overline{z+e}^{\pi} \times \overline{z+e+1}^{\pi} \ldots \overline{z+e+n-1}^{\pi} \times \overline{rz+f}^{-\rho} \times \overline{rz+f+r}^{-\rho} \times \overline{rz+f+2r}^{-\rho} \ldots \overline{rz+f+\overline{m-1}\,r}^{-\rho} \times \overline{z+g}^{-\sigma} \times \overline{z+g+1}^{-\sigma} \times \ldots \overline{z+g+l-1}^{-\sigma} \times$ &c. $\times\, (az^{b}+bz^{b-1}+cz^{b-2}+$&c.$)$. If it be required to reduce the term $\overline{rz+f}^{-\rho}$, &c. to a conformity with the reſt, for $\overline{rz+f}^{-\rho}$, &c. ſubſtitute $\overline{z+\frac{f}{r}}^{-\rho} \times r^{-\rho}$, &c. and it is done. Aſſume for the integral or ſum the quantity $S=\overline{z+e}^{\pi} . \overline{z+e+1}^{\pi} \ldots \overline{z+e+n-2}^{\pi} \times \overline{rz+f}^{-\rho} . \overline{rz+r+f}^{-\rho} \ldots \overline{rz+\overline{m-2}\,r+f}^{-\rho} \times \overline{z+g}^{-\sigma} \times \overline{z+g+1}^{-\sigma} \ldots \overline{z+g+l-2}^{-\sigma} \times$ &c. $\times\, (\alpha z^{b'}+\beta z^{b'-1}+$&c.$)=S$, find its ſucceſſive ſum by writing $z+1$ for z in the ſum S, and let the quantity reſulting be S'; then will the general term be $S-S'$, which equate to the given general term, that is, their correſpondent quantities; and thence may be deduced the index b' and co-efficients α, β, &c.; and conſequently the ſum ſought. If the ſeries does not terminate, then the ſum will be expreſſed by a ſeries proceeding *in infinitum*, according to the reciprocal dimenſions of z.

From $\pi+\rho+\sigma+$ &c. -1 independent integrals of the above-mentioned kind can be deduced the integrals of all quantities of the ſame kind; that is, where b is any whole affirmative number whatever, and the co-efficients a, b, c, &c. are any how varied.

If any factor $z+g$ in the denominator, &c. has no other $z+g+l-1$, which differs from it by a whole number $l-1$; or the factor $rz+f$ has no correſpondent factor $rz+f+mr$, where m is a whole number; then the integral of the above-mentioned ſeries cannot be expreſſed in finite terms of the quantity z. In like manner, if the dimenſions of z in the numerator are leſs than its dimenſions in the denominator by unity, then the integral of the general term cannot be expreſſed by a finite algebraical function of z. If the number of terms to be added be infinite, it is well known that the ſum in this caſe will be infinite.

It may be obſerved, that in finding the ſum of a ſeries, whoſe general term is given, all common diviſors of the numerator and denominator muſt be rejected, otherwiſe ſerieſes may appear difficult to be ſummed, which are very eaſy: for example, let the ſeries be $\frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{4}{4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} + \frac{9}{7 \cdot 8 \cdot 9 \cdot 10 \cdot 11}$ + &c. $= \frac{1}{3}\left(\frac{1}{1 \cdot 2 \cdot 4 \cdot 5} + \frac{2}{4 \cdot 5 \cdot 7 \cdot 8} + \frac{3}{7 \cdot 8 \cdot 10 \cdot 11}\right.$ + &c.), whoſe general term is $\frac{z+1}{3z+1 \cdot 3z+4 \times 3z+2 \cdot 3z+5}$; and by aſſuming, as is before taught, $\overline{3z+1}^{-1} \times \overline{3z+2}^{-1} \times \alpha$ for the ſum ſought; and finding its general term $\overline{3z+1}^{-1} \times \overline{3z+4}^{-1} \times \overline{3z+2}^{-1} \times \overline{3z+5}^{-1} \times 18\,\overline{z+1} \times \alpha$, which equating to the general term given, there reſults $18\alpha = 1$, and the ſum ſought $= \frac{1}{18} \times \frac{1}{3z+1 \cdot 3z+2}$.

Ex. 2. Let the ſeries be $\frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{14}{5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} + \frac{55}{9 \cdot 10 \cdot 11 \cdot 12 \cdot 13} + \frac{140}{13 \cdot 14 \cdot 15 \cdot 16 \cdot 17}$ + &c. $= \frac{1}{24}\left(\frac{1}{1 \cdot 5} + \frac{1}{5 \cdot 9} + \frac{1}{9 \cdot 13} + \frac{1}{13 \cdot 17}\right.$ + &c.), of which the general term is $\frac{1}{24} \times \frac{1}{4z+1 \cdot 4z+5}$; and conſequently the ſum deduced is $\frac{1}{24} \times \frac{1}{4} \times \frac{1}{4z+1}$.

Theſe are ſerieſes given by Mr. De Moivre, and eſteemed by Dr. Taylor *altioris indaginis*.

Some other writers have made ſome ſerieſes to appear more difficult to be ſummed, by not reducing them to their loweſt terms.

7. Having given the principles of a general method of finding the ſum of a ſeries, when its general term can be expreſſed by algebraical, and not exponential, functions of z, the diſtance from the firſt term of the ſeries; it remains to perform the ſame when exponentials are included.

1. Let S the ſum be any algebraical function of z multiplied into $e^{z}=x^{z}$; then will the general term be $Se^{z}-eS'e^{z}=(S-eS')\,e^{z}$; whence, from the general term Te^{z} being given, aſſume quantities in the ſame manner (with the ſame denominator, &c.) as when no exponential was involved, which multiplied into e^{z}, ſuppoſe to be the ſum; from the ſum find its general term, and equate it to the given one by equating their correſpondent co-efficients, and it is done.

Ex. Let the general term be $\frac{z+2}{2z+1\cdot 2z+3}\times e^{z+1}$: aſſume for the ſum ſought $\frac{\alpha}{2z+1}\times e^{z+1}$, whence the general term is $\left(\frac{\alpha}{2z+1}-\frac{\alpha e}{2z+3}\right)e^{z+1}=\frac{2\alpha(1-e)z+3\alpha-\alpha e}{2z+1\cdot 2z+3}\times e^{z+1}$; equate it to the given term, and there reſults $2\alpha(1-e)=1$ and $3\alpha-\alpha e=2$, and conſequently $e=\frac{1}{3}$ and $\alpha=\frac{3}{4}$, if the ſeries can be ſummed.

The ſame obſervation, *viz.* that if any factor in the denominator or irrational quantity have no other correſpondent to it; for example, if the factor be $z+g$, and there is no correſpondent one $x+g+n$, where n is a whole number, then its integral cannot be expreſſed by a finite algebraical function of z.

In the ſame manner may the ſums be found, when the terms are exponentials of ſuperior orders; for the exponential, irrational, &c. quantities in the denominators of the ſums may be eaſily deduced from the preceding principles; and thence, by proceeding as is before taught, the ſum required.

The principles of all theſe caſes have been given in the Meditationes.

8. Mr. James Bernoulli found ſummable ſerieſes by aſſuming a ſeries V, whoſe terms at an infinite diſtance are infinitely little, and ſubtracting the ſeries diminiſhed by any number (l) of terms from the ſeries itſelf, &c.

It is obſerved in the Meditationes, that if $T(m)$, $T(m+n)$, $T(m+n+n')$, $T(m+n+n'+n'')$, &c. be the terms at m, $m+n$, $m+n+n'$, $m+n+n'+n''$, &c. diſtances from the firſt, and $aT(m)+bT(m+n)+cT(m+n+n')+dT(m+n+n'+n'')+$&c. be the general term, it will be ſummable, when $a+b+c+d+$&c.$=0$; the ſum of the ſeries will be $a(T(m)+T(m+1)+T(m+2)+\ldots+T(m+n+n'+n''+\text{\&c.}-1))+b(T(m+n)+T(m+n+1)+T(m+n+2)+\ldots+T(m+n+n'+n''+\text{\&c.}-1))+c(T(m+n+n')+T(m+n+n'+1)+\ldots(T(m+n+n'+n''+\text{\&c.}-1))+\text{\&c.}=H$. If the ſum $a+b+c+d+$&c. be not$=0$, and the ſeries $T(m)+T(m+1)+T(m+2)+$&c. *in infinitum* be a converging one$=S$, then will the ſum of the reſulting ſeries be $(a+b+c+d+\text{\&c.})S-(b+c+d+\text{\&c.})(T^{m}\ldots+T^{m+n-1})-(c+d+\text{\&c.})(T^{m+n}\ldots T^{m+n+n'-1})-(d+\text{\&c.})(T^{m+n+n'}+\ldots T^{m+n+n'+n''-1})-$&c.

8. 2. Let the ſeries V conſiſt of terms, which have only one factor in the denominator, and its numerator$=1$; that is, let the general term be $\frac{1}{rz+e}$, and the ſeries conſequently $\frac{1}{e}+\frac{1}{r+e}+\frac{1}{2r+e}+$&c.$=V$; from the before-mentioned addition or ſubtraction there follows $\frac{a}{rz+e}+\frac{b}{rz+r+e}+\frac{c}{rz+2r+e}+$&c.$=$

$\frac{\alpha z^{m} + \beta z^{m-1} + \gamma z^{m-2} + \&c.}{rz+e \cdot rz+r+e \cdot rz+2r+e \cdot \&c.}$; where m is not greater than the number (N) of factors in the denominator diminiſhed by unity. From α, β, γ, &c. r and e being given, eaſily can be acquired by ſimple equations, or known theorems, the required co-efficients a, b, c, &c. If $m = \text{N} - 1$ and α and $a+b+c+d+\&c. = 0$, then the ſum of the ſeries reſulting will be finite.

8. 3. If the terms of the ſeries aſſumed $\frac{1}{e} - \frac{1}{r+e} + \frac{1}{2r+e} - \frac{1}{3r+e} + \&c.$ be alternately affirmative and negative; then by the preceding caſe find $\frac{\alpha z^{m} + \beta z^{m-1} + \gamma z^{m-2} + \&c.}{rz+e \cdot rz+r+e \cdot rz+2r+e + \&c.} = \frac{a}{rz+e} + \frac{b}{rz+r+e} + \frac{c}{rz+2r+e} + \&c.$ Where the terms of the reſulting ſeries are alternately affirmative and negative, let the two ſubſequent terms be ſuppoſed $\frac{\alpha z^{m} + \beta z^{m-1} + \gamma z^{m-2} + \&c.}{rz+e \cdot rz+r+e \ldots rz+\overline{n-1}r+e} = \frac{a}{rz+e} + \frac{b}{rz+r+e} + \&c.$ and $\frac{\alpha \overline{z+1}^{m} + \beta \overline{z+1}^{m-1} + \gamma \overline{z+1}^{m-2} + \&c.}{rz+r+e \cdot rz+2r+e \ldots rz+nr+e} = \frac{a}{rz+r+e} + \frac{b}{rz+2r+e} + \&c.$ of which the one is affirmative and the other negative: reduce the reſulting ſeries to an affirmative one by ſubtracting the ſubſequent term from its preceding, and it becomes $\frac{(rz+nr+e)(\alpha z^{m} + \beta z^{m-1} + \&c.) - (rz+e)(\alpha \overline{z+1}^{m} + \beta \overline{z+1}^{m-1} + \&c.)}{rz+e \cdot rz+r+e \cdot rz+2r+e \ldots rz+rn+e}$ $= \frac{\overline{n-m}r\alpha z^{m} + \&c.}{rz+e \cdot rz+r+e \ldots rz+rn+e} = \frac{a}{rz+e} + \frac{b-a}{rz+r+e} + \&c$ In this caſe, ſince two terms are added into one, the diſtance from the firſt term of the ſeries will be $\frac{z}{2}$, which ſuppoſe $= w$; and write $2w$ for z in the above-mentioned term, and there reſults $\frac{\overline{n-m}r\alpha z^{m} + \&c.}{rz+e \cdot rz+r+e \ldots rz+nr+e} = \frac{\overline{n-m}r\alpha \times 2^{m} w^{m} + \&c.}{2rw+e \cdot 2rw+r+e \ldots 2rw+nr+e} =$

$\frac{a}{2rw+e}+\frac{b-a}{2rw+r+e}+$&c.; whence the ſum of any ſeries, whoſe general term is $\frac{a'w^m+b'w^{m-1}+\&c.}{2rw+e\,.\,2rw+r+e\,.\,.\,.\,2rw+nr+e}$, where m is a whole number leſs than n by two or more, and w the diſtance from the firſt term of the ſeries can be found from the ſum of the ſeries $\frac{1}{e}-\frac{1}{r+e}+\frac{1}{2r+e}-\frac{1}{3r+e}+$&c.

9. Let there be two ſerieſes $\frac{1}{e}+\frac{1}{e+r}+\frac{1}{e+2r}+$ &c. $=$ S and $\frac{1}{f}+\frac{1}{f+r}+\frac{1}{f+2r}+\frac{1}{f+3r}+$&c.$=$S′, whoſe general terms are reſpectively $+\frac{1}{e+rz}$ and $+\frac{1}{f+rz}$; then from the ſum of theſe two ſerieſes can be collected the ſum of any ſeries, whoſe general term is

$$\frac{\alpha z^m+\beta z^{m-1}+\&c.}{rz+e\,.\,rz+e+r\,.\,rz+e+2r\,.\,.\,.\,rz+\overline{n-1}r+e\times rz+f\,.\,rz+r+f\,.\,.\,.\,rz+f+\overline{m-1}r}$$

$$=\frac{a}{rz+e}+\frac{b}{rz+e+r}+\frac{c}{rz+e+2r}\,.\,.\,.+\frac{\lambda}{rz+\overline{n-1}r+e}+\frac{a'}{rz+f}+\frac{b'}{rz+r+f}$$

$+\frac{c'}{rz+2r+f}\,.\,.\,.+\frac{\mu'}{rz+\overline{m-1}r+f}$; where $e-f$ is not a whole number. Let $a+b+c\,.\,.+\lambda=0$, and $a'+b'+c'\,.\,.\,.+\mu'=0$, then the ſum will be $a\left(\frac{1}{rz+e}+\frac{1}{rz+r+e}\,.\,.\,.+\frac{1}{rz+\overline{n-2}r+e}\right)+b\left(\frac{1}{rz+e+r}\right.$ $\left.+\frac{1}{rz+e+2r}\,.\,.\,.+\frac{1}{rz+\overline{n-2}r+e}\right)+c\left(\frac{1}{rz+e+2r}+\frac{1}{rz+e+3r}+\,.\,.+\right.$ $\left.\frac{1}{rz+\overline{n-2}r+e}\right)+$ &c. $+a'\left(\frac{1}{rz+f}+\frac{1}{rz+r+f}\,.\,.\,.\,\frac{1}{rz+\overline{m-2}r+f}\right)+b'$ $\left(\frac{1}{rz+r+f}+\,.\,.\,.+\frac{1}{rz+\overline{m-2}r+f}\right)+$&c.

2. If the ſerieſes are $\frac{1}{e}-\frac{1}{e+r}+\frac{1}{e+2r}-$&c. and $\frac{1}{f}-\frac{1}{f+r}+\frac{1}{f+2r}$ $-$ &c.; then from the ſum of theſe two ſerieſes can be collected by the principles given above the ſum of any ſeries, whoſe general term is

$$\frac{\alpha z^{m}+\beta z^{m-1}+\gamma z^{m-2}+\&c.}{2rz+e\,.\,2rz+r+e\,.\,2rz+2r+e\ldots 2rz+\overline{n-1}r+e\times 2rz+f\,.\,2rz+r+f\,.\,2rz+2r+f\ldots 2rz+\overline{m-1}r+f}$$

The ſame principle may be applied to find the ſum of any ſeries of the abovementioned ſort, in whoſe denominator are contained other factors, $rz+g$, $rz+g+r$, &c. &c.; or $2rz+g$, $2rz+g+r$, $2rz+g+2r$, &c. Like propoſitions may be deduced from ſerieſes, in which r and r', &c. and the factors $rz+e$ and $r'z+g$, &c. denote different quantities.

10. An apparently more general method may be given from aſſuming a ſeries or ſerieſes as before; and adding every two, three, four, &c. (n) ſucceſſive terms together for terms of a new ſeries beginning from the firſt, ſecond, third, &c. n^{th} term; and in general adding together two, three, &c. n ſucceſſive general terms; and in their ſum writing for z the diſtance from the firſt term of the ſeries $2z+a$, $3z+a$, &c. $nz+a$; there will reſult the general term of a ſeries not to be found from the above-mentioned addition.

Ex. Let the ſeries aſſumed be $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+$ &c. *in infinitum*, of which the general term beginning from the firſt is $\frac{1}{z+1}$; add three ſucceſſive general terms $\frac{1}{z+1}+\frac{1}{z+2}+\frac{1}{z+3}$ $=\frac{3z^{2}+12z+11}{z+1\,.\,z+2\,.\,z+3}$; in this term for z write $3z$, and there reſults $\frac{27z^{2}+36z+11}{3z+1\,.\,3z+2\,.\,3z+3}$. In the ſame manner, if the beginning is inſtituted from the ſecond or third term of the given ſeries, the terms reſulting will be $\frac{3z^{2}+18z+26}{z+2\,.\,z+3\,.\,z+4}$ and $\frac{3z^{2}+24z+47}{z+3\,.\,z+4\,.\,z+5}$. In theſe terms for z write $3z$, and there reſult $\frac{27z^{2}+54z+26}{3z+2\,.\,3z+3\,.\,3z+4}$ and $\frac{27z^{2}+72z+47}{3z+3\,.\,3z+4\,.\,3z+5}$.

If the terms of the given ſeries are alternately affirmative and negative, the terms of the reſulting ſeries will be alternately affirmative and negative, if n be an odd number; otherwiſe its terms will be all affirmative. The ſum of this ſeries will be finite or infinite, as the ſum of the ſeries $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c.$ is finite or infinite; but from it, by the preceding method of addition or ſubtraction of Mr. BERNOULLI's, or a like method applied to more ſerieſes, may be found the ſums of different finite ſerieſes.

It may be obſerved, that from Mr. BERNOULLI's addition or ſubtraction can never be deduced the ſerieſes which ariſe from this method; for, by his method, the denominator can never have any factors but what are contained in the denominators of the given ſeries, *viz.* (in the ſeries $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \&c.$), $z + l$, where l is a whole number; but by this method are introduced into the denominator the factors $2z + l$, $3z + l$, &c. and $nz + l$, or which may be reduced to the ſame $\left(z + \frac{l}{n}\right) \times n$.

If n ſucceſſive general terms of the ſerieſes ariſing from Mr. BERNOULLI's addition or ſubtraction be added together, and in the quantity thence ariſing for z the diſtance from the firſt term of the ſeries be ſubſtituted nz, there will be produced ſerieſes of the above-mentioned formula.

11. Multiply two converging ſerieſes $a + bx + cx^2 + dx^3 + \&c. = S$ and $\alpha + \beta x + \gamma x^2 + \&c. = V$, or find any rational and integral function of them, and the ſeries reſulting will be finite and $= S \times V$, &c. Let $\alpha + \beta x + \gamma x^2 + \&c.\ x^m = V$ be finite, and the reſulting ſeries will be finite and $= S \times V$, &c. If S be a ſeries converging or not, whoſe ultimate terms are leſs than any finite quantity, then will the ſeries $(a + bx + cx^2 + \&c.) \times (\alpha + \beta x + \gamma x^2 + \&c.\ x^m) = V \times S$ be a converging one, if $\alpha + \beta x + \gamma x^2 + \ldots \&c.\ x^m = 0$; which caſe was given by Mr. DE MOIVRE.

Mr. BERNOULLI's addition, &c. can be applied to ſerieſes of this kind. For example, let the given ſeries be $\frac{1}{e}+\frac{1}{e+1}x+\frac{1}{e+2}x^2+\&c.=S$. From this ſeries ſubtract the ſame ſeries diminiſhed by m terms, *viz.* $\frac{1}{e+m}x^m+\frac{1}{e+m+1}x^{m+1}+\frac{1}{e+m+2}x^{m+2}+\&c.$ and there remains $\frac{e+m-ex^m}{e\,.\,m+e}+\frac{e+m+1-\overline{e+1}x^m}{e+1\,.\,e+m+1}x+\frac{e+m+2-\overline{e+2}x^m}{e+2\,.\,e+m+2}x^2+\frac{e+m+3-\overline{e+3}x^m}{e+3\,.\,e+m+3}\times x^3+\&c.$; for x^m write A, then will the ſeries become $\frac{m-eA}{e\,.\,m+e}+\frac{e+m+1-\overline{e+1}A}{e+1\,.\,e+m+1}x+\frac{e+m+2-\overline{e+2}A}{e+2\,.\,e+m+2}x^2+\frac{e+m+3-\overline{e+3}A}{e+3\,.\,e+m+3}x^3+\&c.=\frac{1}{e}+\frac{1}{e+1}x+\frac{1}{e+2}x^2\ldots\frac{1}{e+m-1}x^{m-1}$.

Let the general term be $\frac{az^m+bz^{m-1}+cz^{m-2}+\&c.}{z+e\,.\,z+e+1\,.\,z+e+2\,.\,.\,z+e+n-1}\times x^z=\left(\frac{\alpha}{z+e}+\frac{\beta}{z+e+1}+\frac{\gamma}{z+e+2}\ldots\frac{X}{z+e+n-1}\right)x^z$. Suppoſe $\beta=\beta' x$, $\gamma=\gamma' x^2$, $\delta=\delta x^3$, $..\ X=X' x^{n-1}$; then will the ſum of the above-mentioned ſeries be $(\alpha+\beta'+\gamma'+\delta'+\&c.)\times S-\frac{1}{e}(\beta'+\gamma'+\delta'+\&c.)-\frac{1}{e+1}(\gamma'+\delta'+\&c.)-\frac{1}{e+2}(\delta'+\&c.)-\&c.$

From the ſum of the ſeries $\frac{1}{e}-\frac{x}{e+1}+\frac{x^2}{e+2}-\&c.$ by theſe and the principles before delivered can be deduced the ſum of any ſeries, whoſe general term is

$$\frac{az^m+bz^{m-1}+\&c.}{2z+e\,.\,2z+e+1\,.\,2z+e+2\,.\,2z+e+3\times\&c.}x^z.$$

In like manner from the ſum of the ſerieſes $\frac{x}{e}+\frac{x^2}{e+1}+\frac{x^3}{e+2}+\&c.$ $\frac{x}{f}+\frac{x^2}{f+1}+\frac{x^3}{f+2}+\&c.$ $\frac{x}{g}+\frac{x^2}{g+1}+\frac{x^3}{g+2}+\&c.$ &c. can be deduced the ſum of any ſeries, whoſe general term is

$$\frac{az^m + bz^{m-1} + \&c.}{z+e \, . \, z+e+1 \, . \, z+e+2 \times \&c. \times z+f \, . \, z+f+1 \, . \, z+f+2 \, . \, \&c. \times z+g \, . \, z+g+1 \, . \, \&c.} \times x^z.$$

And alſo from the ſum of the ſerieſes $\frac{1}{e} - \frac{1}{e+1}x + \frac{x^2}{e+2} - \&c.$ $\frac{1}{f} - \frac{x}{f+1} + \frac{x^2}{f+2} - \&c.$ $\frac{1}{g} - \frac{x}{g+1} + \frac{x^2}{g+2} - \&c.$ &c. can be deduced the ſum of any ſeries, whoſe general term is

$$\frac{az^m + bz^{m-1} + \&c.}{2z+e \, . \, 2z+e+1 \, . \, \&c. \times 2z+f \, . \, 2z+f+1 \, . \, \&c. \; 2z+g \, . \, 2z+g+1 \, . \, \&c.} \times x^z.$$

The method of adding more terms of a given ſeries together, as before taught, may be applied to theſe and all other ſerieſes. For example: let the given ſeries be $1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \&c.$; add two terms conſtantly together, and it becomes $1 + \frac{1}{2}x + \&c. = \frac{2+x}{2} + \frac{4+3x}{3 \cdot 4}x^2 + \frac{6+5x}{5 \cdot 6}x^4 + \&c. = \frac{2+A}{2} + \frac{4+3A}{3 \cdot 4}x^2 + \frac{6+5A}{5 \cdot 6}x^4 + \&c.$ whence the general term is $\frac{2z+2+(2z+1)}{2z+2}$ $\frac{x}{2z+1}x^{2z}$. From the methods before given of addition, ſubtraction, and multiplication; and the ſerieſes found by this method, can be derived ſerieſes, whoſe ſums are known.

12. Suppoſe a given ſeries $ax^n + bx^{n \pm s} + cx^{n \pm 2s} + dx^{n \pm 3s} + \&c.$ whoſe ſum p is either an algebraical, exponential, or fluential fluxion of x; multiply the equation $p = ax^n + bx^{n \pm s} + cx^{n \pm 2s} + dx^{n \pm 3s} + \&c.$ into $x^{\pm r - n}$, and there reſults $ax^{\pm r - n}p = ax^{\pm r} + bx^{\pm r \pm s} + cx^{\pm r \pm 2s} + \&c.$; find the fluxion of this equation, and there follows $\frac{1}{\dot{x}}$ multiplied into the fluxion of the quantity $(x^{\pm r - n}p)$ $= \pm rax^{\pm r - 1} + (\pm r \pm s)\, bx^{\pm r \pm s - 1} + (\pm r \pm 2s)\, cx^{\pm r \pm 2s - 1} + \&c.$ of which the general term is $(\pm r \pm zs) \times t$, where z denotes the diſtance from the firſt term of the ſeries, and t

is the term in the given ſeries, whoſe diſtance from the firſt is z. In the ſame manner may be deduced the ſum of a ſeries whoſe general term is $t' \times \overline{\pm r \pm zs} \times \overline{\pm r' \pm z \pm ns}$, or by repeated operations $t' \times \overline{ez^2 + fz + g}$, where t' is a term of the given equation, whoſe diſtance from the firſt term is z. And in general, from the ſum of a given ſeries, whoſe fluxion can be found, and whoſe general term is t', can be deduced by continued multiplication, and finding the fluxion, the ſum of a ſeries or quantity, of which the general term is At', where A is any function of the following kind $a'z^m + b'z^{m-1} + c'z^{m-2} + \&c.$ in which z denotes the diſtance from the firſt term of the ſeries, and m a whole number. It is to be obſerved, that if the given ſeries converges in a ratio, which is at leaſt equal to the ratio of the convergency of ſome geometrical ſeries, the reſulting equation will always converge. But if in a leſs ratio, then it will ſometimes converge, ſometimes not, according to the ratio which the ſucceſſive terms of the reſulting ſeries have to each other at an infinite diſtance.

Corollary. $\frac{p \cdot p+1 \cdot p+2 \cdot p+3 \cdot \cdot p+z}{r \cdot r+1 \cdot r+2 \cdot r+3 \cdot \cdot r+z} = \frac{p+z \cdot p+z-1 \cdot p+z-2 \cdot p+z-3 \cdot \cdot \cdot z+r+1}{r \cdot r+1 \cdot r+2 \cdot r+3 \cdot \cdot \cdot \cdot \cdot \cdot p-1}$, if $p-r$ be a whole affirmative number; but this latter quantity has the formula above-mentioned $az^m + bz^{m-2} + cz^{m-3} + \&c.$; and conſequently, if the ſum of the ſeries $a + bx^s + cx^{2s} + dx^{3s} + \&c. = p$ be known, by this method can be deduced the ſum of the ſeries $a + \frac{p}{r} bx^s + \frac{p \cdot p+1}{r \cdot r+1} cx^{2s} + \frac{p \cdot p+1 \cdot p+2}{r \cdot r+1 \cdot r+2} dx^{3s} + \&c.$

Ex. 1. Since $\overline{a+x}^{\frac{m}{n}} = a^{\frac{m}{n}} \left(1 + \frac{m}{n} \times \frac{x}{a} + \frac{m}{n} \times \frac{m-n}{2n} a^{-2}x^2 + \frac{m}{n} \cdot \frac{m-n}{2n} \cdot \frac{m-2n}{3n} a^{-3}x^3 + \&c.\right)$; multiply the ſucceſſive terms of this ſeries into the terms of the ſeries 1, $\frac{p}{r}$, $\frac{p \cdot p+1}{r \cdot r+1}$, &c. and a ſeries is deduced $a^{\frac{m}{n}} + \frac{p \cdot m}{r \cdot n} a^{+\frac{m}{n}-1} x + \frac{p \cdot p+1 \times m \cdot m-n}{r \cdot r+1 \cdot n \cdot 2n} x^{+\frac{m}{n}-2}$ + &c. whoſe ſum is known, if the ſum of the ſeries $= \overline{a+x}^{+\frac{m}{n}}$ is known.

Ex. 2. If the ſeries begins from the $l+1^{\text{th}}$ term of the above-mentioned binomial theorem $a\frac{m}{n} + \frac{m}{n}\, a^{+\frac{m}{n}-1} x +$ &c. *viz.* the ſeries be $H \times \overline{1 + \frac{m-\overline{l+1}\,n}{l+2n}\frac{x}{a} + \frac{m-\overline{l+2}\,n}{l+3n}\frac{x^2}{a^2} + \frac{m-\overline{l+3}\,n}{l+4n}\frac{x^3}{a^3} + \text{\&c.}}$ of which let the reſpective terms be multiplied into 1, $\frac{p}{r}$, $\frac{p \cdot p+1}{r \cdot r+1}$, &c. there will reſult a ſeries whoſe ſum is known.

Ex. 3. From the rule firſt given by me for finding the ſum of the terms at b diſtances from each other, the ſum of the ſeries $1 + \frac{m-\overline{l+1}\,n}{\overline{l+2} \cdot n} \times \frac{m-\overline{l+2}\,n}{\overline{l+3}\,n} \cdot\cdot \frac{m-\overline{l+b}\,n}{\overline{l+b+1}\,n} \times \frac{x^b}{a^b} + P \times \frac{m-\overline{l+b+1}\,n}{\overline{l+b+2}\,n} \times] \frac{m-\overline{l+b+2}\,n}{\overline{l+b+3}\,n} \cdots \frac{m-\overline{l+2b}\,n}{\overline{l+2b+1}\,n} \frac{x^{2b}}{a^{2b}} +$ &c. where P denotes the co-efficient of the preceding term, can be deduced; and conſequently the ſum of the ſeries deduced from multiplying the ſucceſſive terms of this ſeries into the quantities 1, $\frac{p}{r}$, $\frac{p \cdot p+1}{r \cdot r+1}$, &c. reſpectively.

The general principles of this caſe were firſt delivered by Mr. BERNOULLI, Mr. DE MOIVRE, Mr. EULER, &c.

12. Aſſume the ſeries $a + bx^n + cx^{2n} +$ &c. $= p$, multiply it into $x^{\alpha-1}\dot{x}$, and find the fluent, then will $\frac{1}{\alpha} x^{\alpha} p - \frac{1}{\alpha}\int x^{\alpha}\dot{p} = \frac{1}{\alpha} a x^{\alpha} +$

$\frac{1}{\alpha+n}bx^{\alpha+n}+\frac{1}{\alpha+2n}cx^{\alpha+2n}+$ &c.; multiply this equation into $x^{\beta-\alpha-1}\dot{x}$, and find the fluent of the equation reſulting, which will be $\frac{1}{\beta}\times\frac{1}{\alpha}x^{\beta}p-\frac{1}{\alpha}\cdot\frac{1}{\beta}\int x^{\beta}\dot{p}-\frac{1}{\alpha}\times\frac{1}{\beta-\alpha}x^{\beta-\alpha}\int x^{\alpha}\dot{p}+\frac{1}{\alpha}\cdot\frac{1}{\beta-\alpha}\int x^{\beta}\dot{p}=\frac{1}{\alpha}\cdot\frac{1}{\beta}ax^{\beta}+\frac{1}{\alpha+n}\cdot\frac{1}{\beta+n}bx^{\beta+n}+\frac{1}{\alpha+2n}\cdot\frac{1}{\beta+2n}cx^{\beta+2n}+$ &c.; divide by x^{β}, and there reſults $\frac{1}{\beta}\cdot\frac{1}{\alpha}p+\frac{1}{\alpha}\cdot\frac{1}{\alpha-\beta}x^{-\alpha}\int x^{\alpha}\dot{p}+\frac{1}{\beta}\cdot\frac{1}{\beta-\alpha}x^{-\beta}\int x^{\beta}\dot{p}=\frac{1}{\alpha}\cdot\frac{1}{\beta}a+\frac{1}{\alpha+n}\cdot\frac{1}{\beta+n}bx^{n}+$ &c.: and in general $\frac{1}{\alpha}\cdot\frac{1}{\beta}\cdot\frac{1}{\gamma}\cdot$ &c. $p+\frac{1}{\alpha}\cdot\frac{1}{\alpha-\beta}\cdot\frac{1}{\alpha-\gamma}x^{-\alpha}\int x^{\alpha}\dot{p}+\frac{1}{\beta}\cdot\frac{1}{\beta-\alpha}\cdot\frac{1}{\beta-\gamma}\cdot$ &c. $x^{-\beta}\int x^{\beta}\dot{p}+\frac{1}{\gamma}\cdot\frac{1}{\gamma-\alpha}\cdot\frac{1}{\gamma-\beta}$, &c. $x^{-\gamma}\int x^{\gamma}\dot{p}=\frac{1}{\alpha}\cdot\frac{1}{\beta}\cdot\frac{1}{\gamma}$ &c. $a+\frac{1}{\alpha+n}\cdot\frac{1}{\beta+n}\cdot\frac{1}{\gamma+n}$ &c. $bx^{n}+\frac{1}{\alpha+2n}\cdot\frac{1}{\beta+2n}\cdot\frac{1}{\gamma+2n}$, &c. $cx^{2n}+$ &c. whence the law of continuation is immediately manifeſt.

Hence, if no two quantities α, β, γ, δ, &c. be equal to each other; and the ſucceſſive terms a, b, c, d, &c. of any ſeries $a+bx^{n}+cx^{2n}+$ &c. $=p$ be divided by $\alpha.\beta.\gamma.\delta.$ &c.; $\overline{a+n}\,.\,\overline{\beta+n}\,.\,\overline{\gamma+n}\,.\,\overline{\delta+n}\,.$ &c.; $\alpha+2n\,.\,\beta+2n\,.\,\gamma+2n\,.\,\delta+2n\,.$ &c. &c.; and in general by $\alpha+nz\,.\,\beta+nz\,.\,\gamma+nz\,.\,\delta+nz\,.$ &c. &c.; then can the ſum of the ſeries be found from the fluents of the fluxions $x^{\alpha}\dot{p}$, $x^{\beta}\dot{p}$, $x^{\gamma}\dot{p}$, $x^{\delta}\dot{p}$, &c. as has been obſerved in the Meditationes. If two are equal, *viz.* $\alpha=\beta$, then alſo the fluent of the fluxion $\frac{\dot{x}}{x}\int x^{\alpha}\dot{p}$ is required. If three are equal *viz.* $\alpha=\beta=\gamma$; then it is neceſſary to find the fluent of the fluxion $\frac{\dot{x}}{x}\int\frac{\dot{x}}{x}\int x^{\alpha}\dot{p}$; and ſo on.

1. Let $\dot{p}=\frac{\dot{x}}{1\pm x^{n}}$; and if the differences of the quantities α, β, γ, δ, &c. are diviſible by n, from the fluent of the fluxion $x^{\alpha}\dot{p}$ can be deduced the fluents of all the other fluxions $x^{\beta}\dot{p}$, $x^{\gamma}\dot{p}$, &c.; and in general, if $\alpha - \beta$ is diviſible by n, then from the fluent of the fluxion $x^{\alpha}\dot{p}$ can be deduced the fluent of the fluxion $x^{\beta}\dot{p}$.

2. Suppoſe $p =$ the terms of the binomial theorem expanded according to the dimenſions of x, *viz.* $(a+bx^n)^{\frac{r}{s}} = a^{\frac{r}{s}} + \frac{r}{s} a^{\frac{r}{s}-1} bx^n + \&c.$ beginning from the firſt or any other terms; then, if α, β, &c. divided by n give whole affirmative numbers, will all the fluxions $x^{\alpha}\dot{p}$, $x^{\beta}\dot{p}$, $x^{\gamma}\dot{p}$, &c. be integrable; and if the differences of the quantities α, β, γ, δ, &c. are diviſible by n, from the fluent of the fluxion $x^{\alpha}\dot{p}$ can be deduced the fluents of the fluxions $x^{\beta}\dot{p}$, $x^{\gamma}\dot{p}$, &c.

If p denotes the ſum of the alternate or terms whoſe diſtance from each other are m, of the binomial theorem, the ſame may be applied.

3. If $p = \overline{a+bx^n+cx^{2n}}^{\frac{r}{s}}$; and α, β, γ, δ, &c. divided by n give whole affirmative numbers, then from $\int x^{\alpha}\dot{p}$ can be deduced all the remainder $\int x^{\beta}\dot{p}$, $\int x^{\gamma}\dot{p}$, &c.: and in general from two can be deduced all the remainder.

To find when the ſum of any ſeries of this kind can be found, add together each of the fluents, which can be found from each other, and not otherwiſe, and ſuppoſe their ſum $= 0$; and ſo of any other ſimilar fluent, and from the reſulting equations can be diſcovered when the ſeries can be integrated.

13. If the general term of a ſeries contains in it more variable quantities, z, v, w, &c.; then find the ſum of the ſeries, firſt, from the hypotheſis that one of them (z) is only varia-

ble, which, properly corrected, let be A; in the quantity A ſuppoſe all the quantities invariable but ſome other v, and find the ſum of the ſeries thence reſulting, which let be B, and ſo on; and the ſum of the ſeries will be deduced.

Ex. Let the term be $\frac{1}{z\cdot\overline{z+1}\times v\cdot\overline{v+1}\cdot\overline{v+2}}$; the dimenſions of z and v, &c. in the denominator muſt be at leaſt greater than its dimenſions in the numerator by a quantity greater than the number of the quantities z, v, &c. which proceed *in infinitum* increaſed by unity. Firſt, ſuppoſe z only variable, and the ſum of the infinite ſeries reſulting will be $\frac{1}{z\cdot v\cdot\overline{v+1}\cdot\overline{v+2}} = \text{A}$; then ſuppoſe v only variable, and the ſum reſulting will be $\frac{1}{2z\cdot v\cdot\overline{v+1}} = \text{B}$, which is the ſum required.

If it be ſuppoſed, that the quantities z and v, &c. in the ſame term ſhall never have the ſame values, then ſuppoſe z and v always to have the ſame values, and the general term $\frac{1}{z\cdot\overline{z+1}\cdot v\cdot\overline{v+1}\cdot\overline{v+2}}$ becomes $\frac{1}{z^2\cdot\overline{z+1}^2\cdot\overline{z+2}}$, of which let the ſum be V, then will B − V be the ſum required.

On this and ſome other ſubjects more have been given in the Meditationes.

14. If the ſum of the ſeries cannot be found in finite terms, and it is neceſſary to recur to infinite ſeries; it is obſerved in the Meditationes to be generally neceſſary to add ſo many terms together, that the diſtance from the firſt term of the ſeries may conſiderably exceed the greateſt root of an equation reſulting from the general term made $= 0$; and afterwards a ſeries more converging may commonly be deduced from the fluents of fluxions reſulting from neglecting all but the greateſt quantities in the general terms reſulting; and by other

different methods. Mr. Nicholas Bernoulli and Mr. Mommort inveſtigated the ſum of the ſeries (P) $A+Br+Cr^2+$ &c. by a ſeries (Q) $\frac{A}{1-r}+\frac{d'r}{(1-r)^2}+\frac{d''r^2}{(1-r)^3}+\frac{d'''r^3}{(1-r)^4}+$ &c.; where d', d'', d''', &c. denote the ſucceſſive differences of the terms A, B, C, D, &c. If r be negative, the denominators become $1+r$, $(1+r)^2$, $(1+r)^3$, &c.

It has been obſerved, in the Meditationes, that in ſwift converging ſeries the ſeries P will converge more ſwiftly than the ſeries Q; in ſeries converging according to a geometrical ratio, ſometimes the one will converge more ſwift, and ſometimes the other. In other ſeries, which converge more ſlow, where moſt commonly r nearly $=1$, it cannot in general be ſaid, which of the ſerieſes will converge the ſwifteſt. The preceding remark, *viz.* the addition of the firſt terms of the ſeries, is neceſſary in moſt caſes of finding the ſums by ſerieſes of this kind.

It is not unworthy of obſervation, that in almoſt all caſes of infinite ſeries, the convergency depends on the roots of the given equations, which remark was firſt publiſhed in the Meditationes. For example: in finding approximates to the roots of given equations the convergency depends on how much the approximates given are more near to one root than to any other; and conſequently, when two or more roots or values of an unknown quantity are nearly equal, different rules are to be applied, which are improvements of the rule of falſe. This rule, and the above-mentioned obſervations were firſt given in the Meditationes Algebraicæ et Analyticæ, with ſeveral other additions on ſimilar ſubjects.

Many more things concerning the ſummation of ſeries, which depend on fluxional, &c. equations, might be added; but I ſhall conclude this paper with congratulating myſelf, that ſome algebraical inventions publiſhed by me have been ſince thought not unworthy of being publiſhed by ſome of the greateſt mathematicians of this or any other age.

1ſt, In the year 1757, I ſent to the Royal Society the firſt edition of my Meditationes Algebraicæ: they were printed and publiſhed in the years 1760 and 1762, with Properties of Curve Lines, under the title of Miſcellanea Analytica, and a copy of them ſent to Mr. EULER in the beginning of the year 1763, in which was contained a reſolution of algebraical equations, not inferior, on account of its generality and facility, to any yet publiſhed (*viz.* $y = a\sqrt[n]{p} + b\sqrt[n]{p^2} + c\sqrt[n]{p^3} + \ldots\sqrt[n]{p^{n-1}}$). This reſolution was publiſhed by Mr. EULER in the Peterſburg Acts for the year 1764. Whether Mr. EULER ever received my book, I cannot pretend to ſay; nor is it material: for the fact is, that it was publiſhed by me in the year 1760 and 1762, and firſt by Mr. EULER in the year 1764. Mr. DE LA GRANGE and Mr. BEZOUT have aſcribed this reſolution to Mr. EULER, as firſt publiſhed in the year 1764, not having ſeen (I ſuppoſe) my Miſcell. Analyt. Mr. BEZOUT found from it ſome new equations, of which the reſolution is known, and applied it to the reduction of equations: more new equations are given, and the reſolution rendered more eaſy by me in the Philoſophical Tranſactions. 2d, In the above-mentioned Miſcell. Analyt. an equation is transformed into another, of which the roots are the ſquares of the differences of the roots of the given equation; and it is aſſerted in that book, that if the

coefficients

co-efficients of the terms of the reſulting equations change continually from + to − and − to +, the roots of the given equation are all poſſible, otherwiſe not; and in a paper, inſerted by me in the Philoſophical Tranſactions for the year 1764, in which is found from this transformation, when there are none, two or four impoſſible roots contained in an algebraical equation of four or five dimenſions; it is obſerved, that there will be none or four, &c. impoſſible roots contained in the given equation, if the laſt term be + or −; and two, &c. on the contrary, if the laſt term be − or +. Theſe obſervations and tranſformation have been ſince publiſhed and explained in the Berlin Acts for the years 1767 and 1768, by Mr. DE LA GRANGE. 3d. In the Miſcell. Anal. an equation is transformed into another, whoſe roots are the ſquares, &c. of the roots of a given equation; and it is aſſerted, that there are at leaſt ſo many impoſſible roots contained in the given equation, as there are continual progreſſes in the reſulting equation from + to + and − to −. It is afterwards remarked, that theſe rules ſometimes find impoſſible roots when Sir ISAAC NEWTON's, and ſuch like rules, fail; and that Sir ISAAC NEWTON's, &c. will find them, when this rule fails. This rule may ſomewhat further be promoted by firſt changing the given equation, whoſe root is x, into another whoſe root is $\sqrt{-1}x$; but, in my opinion, the rule of HARRIOT's, which only finds whether there are impoſſible roots contained in a cubic equation or not, is to be preferred to theſe rules, which, in equations of any dimenſions, of which the impoſſible roots cannot generally be found from the rules, ſeldom find the true number. 4th, It is remarked, that rules which diſcover the true number of impoſſible roots require immenſe calculations, ſince they muſt neceſſarily find, when the roots become equal. In order to this, in the Miſcell. Anal. there is found an equation, whoſe roots are the reciprocals of the differences of any two roots of the given equation; and from finding a quantity (π) greater than the greateſt root of the given, and $\left(\frac{1}{A}\right)$ greater than the greateſt root of the reſulting equation, and ſubſtituting π, $\pi - A$, $\pi - 2A$, &c. for x in the given equation; will always be found the true number of impoſſible roots. 5th, In the ſame book are aſſumed two equations $(nx^{n-1} - \overline{n-1}px^{n-2} + \overline{n-2}qx^{n-3} - \&c. = 0$ and $x^n - px^{n-1} + \&c. = w)$, and thence deduced an equation, whoſe root is w, from which, in ſome caſes, can be found the number of impoſſible roots.

6. In the Miſcell. Anal. is given the law of a ſeries, and its demonſtration, which finds the ſum of the powers of the roots of a given equation from its co-efficients. Mr. EULER has ſince publiſhed the ſame in the Peterſburg Acts. Mr. DE LA GRANGE printed a property of this ſeries, alſo printed by me about the ſame time; *viz.* that if the ſeries was continued *in infinitum*, the powers would obſerve the ſame law as the roots, which indeed immediately follows from the ſeries itſelf; but from thence with the greateſt ſagacity he deduces the law of the reverſion of the ſeries $(y = a + bx + cx^2 + dx^3 + \&c.)$: it has ſince been given in a different manner from ſimilar principles in the Medit. Analyt. 7. In the Miſcell. Analyt. the law of a ſeries is given for finding the ſum of all quantities of this kind $(\alpha^m \times \beta^n \times \gamma^r \times \delta^s \times \&c. + \&c.)$ where α, β, γ, δ, &c. denote the roots of a given equation, from the powers of the roots of the given equation. This law, with a different notation, has been ſince publiſhed in the Paris Acts by Mr. VANDERMONDE; who indeed mentions that he

had heard, that a ſeries for that purpoſe was contained in my book, but had not ſeen it. In the ſame book is given a method of finding the aggregates of any algebraical functions of each of the roots of given equations, which is ſomewhat improved in the latter editions. 8. In the ſame book are aſſumed $\frac{az^{n}+bz^{n-1}+\&c.}{pz^{m}+qz^{m-1}+\&c.}$ and $\frac{Az^{n'}+Bz^{n'-1}+\&c.}{pz^{m}+qz^{m-1}+\&c.}$, where z is any rational quantity whatever for x and y, the unknown quantities of a given equation of two or more dimenſions. 9. In the Miſcell. Analyt. a biquadratic ($x^4+2px^3=qx^2+rx+s$, of which no term is deſtroyed) is reduced to a quadratic ($x^2+px+n=\sqrt{p^2+2n+q}\,x+\sqrt{s+n^2}$;) and in the ſecond edition of it, printed in the years 1767, 1768, 1769, and publiſhed in the beginning of the year 1770, the values of n are found $\frac{\alpha\beta+\gamma\delta}{2}$, $\frac{\alpha\gamma+\beta\delta}{2}$, and $\frac{\alpha\delta+\beta\gamma}{2}$; and the ſix values of $\sqrt{y^2+2n+q}$ reſpectively $\frac{\alpha+\beta-\gamma-\delta}{2}$, $\frac{\alpha+\gamma-\beta-\delta}{2}$, $\frac{\alpha+\delta-\beta-\gamma}{2}$, and their negatives; and the ſix values of $\sqrt{s+n^2}$ reſpectively $\frac{\alpha\beta-\gamma\delta}{2}$, $\frac{\alpha\gamma-\beta\delta}{2}$, $\frac{\alpha\delta-\beta\gamma}{2}$, and their negatives. 10. From a given biquadratic ($y^4+qy^2+ry+s=0$) by aſſuming $y^2+ay+b=v$ and a and b ſuch quantities as to make the ſecond and fourth terms of the reſulting equations to vaniſh, there reſults an equation ($v^4+Av^2+B=0$) of the formula of a quadratic. Mr. DE LA GRANGE has aſcribed this reſolution to Mr. TSCHIRNHAUSEN; but in the Leipſic Acts the reſolution of a cubic is given by Mr. TSCHIRNHAUSEN, but not of a biquadratic: his general deſign ſeems to be the extermination of all the terms.

11. Mr. Euler or Mr. de la Grange found, that if α be a root of the equation $x^n - 1 = 0$, where n is a prime number, $\alpha, \alpha^2, \alpha^3, \ldots \alpha^{n-1}$, 1 will be ($n$) roots of it. More on a ſimilar ſubject has been added in the laſt edition of the Medit. Algebr. 12. It is obſerved in the Miſcell. Analyt. that Cardan's or Scipio Ferreus's reſolution of a cubic is a reſolution of three different cubic equations; and in the Medit. Algeb. 1770, the three cubics are given, and the rationale of the reſolution (for example: if α, β, and γ, be the roots of the cubic equation $x^3 + qx - r = 0$, then is given the function of the above roots, which are the roots of the reducing equation $z^6 - rz^3 = q^3$); and alſo the rationale of the common reſolution of biquadratics. 13. It is aſſerted in the Miſcell. that if the terms ($My^n + by^{n-1}x + cy^{n-2}x^2 + \&c.$ and $Ny^m + By^{m-1}x + Cy^{m-2}x^2 + \&c.$) of two equations of n and m dimenſions, which contain the greateſt dimenſions of x and y have a common diviſor, the equation whoſe root is x or y, will not aſcend to $n \times m$ dimenſions; and if the equation, whoſe root is x or y, aſcends to $n \times m$ dimenſions, the ſum of its roots depends on the terms of n and $n - 1$ dimenſions in the one, and m and $m - 1$ dimenſions in the other equation, &c. It is alſo aſſerted, in the Miſcell. that if three algebraical equations of n, m, and r dimenſions contain three unknown quantities x, y, and z, the equation, whoſe root is x or y or z, cannot aſcend to more than $n \,.\, m \,.\, r$ dimenſions. 14. Mr. Bezout has given two very elegant propoſitions for finding the dimenſions of the equation whoſe root is x or y, &c ; where x, y, &c. are unknown quantities contained in two or more (*b*) algebraical equations of π, ρ, σ, &c. dimenſions, and in which ſome of the unknown quantities do not aſcend to the above π, ρ, σ, &c.

dimenſions reſpectively. In demonſtrating theſe propoſitions he uſes one (amongſt others) before given by me (*viz.* if an equation of n dimenſions contains m unknown quantities, the number of different terms which may be contained in it will be $\overline{n+1} \cdot \frac{n+2}{2} \cdot \frac{n+3}{3} \cdot \cdot \frac{n+m}{m})$. In the Medit. 1770 there is given a method of finding in many caſes the dimenſions of the equation, whoſe root is x or y, &c.; from which one, if not both, of the above-mentioned caſes may more eaſily be deduced, and others added. 15. In the Medit. 1770 is obſerved, that if there be n equations containing m unknown quantities, where n is greater than m, there will be $n-m$ equations of conditions, &c. 16. In the Miſcell. is given and demonſtrated the ſubſequent propoſition; *viz.* if two equations contain two unknown quantities x and y, in which x and y are ſimilarly involved; the equation, whoſe root is x or y will have twice the number of roots which the equation, whoſe root is $x+y$, x^2+y^2, &c. has. In the Medit. 1770 the ſame reaſoning is applied to equations, which have two, three, four, &c. quantities ſimilarly involved. 17. Mr. DE LA GRANGE has done me the honour to demonſtrate my method of finding the number of affirmative and negative roots contained in a biquadratic equation. A demonſtration of my rule for finding the number of affirmative, negative, and impoſſible roots contained in the equation $x^n + Ax^m + B = 0$ is alſo omitted, on account of its eaſe and length. From the Medit. the inveſtigation of finding the true number of affirmative and negative roots appears to be as difficult a problem as the finding the true number of impoſſible roots; and it further appears, that the common methods in both caſes can ſeldom be depended on. But their faults lie on different ſides, the one generally finds too many, the other too few. 18. In the Medit. 1770, from the number of impoſſible roots in a given equation $(x^n - px^{n-1} + \&c. = 0)$ is found the number of impoſſible roots in an equation, whoſe roots (v) have any aſſignable relation to the roots of a given equation; and examples are given in the relation $(nx^{n-1} - \overline{n-1}px^{n-2} + \&c. = v)$; and in an equation, whoſe roots are the ſquares of the differences of the roots of the given equation. 19. It is obſerved in the Medit. 1770, that in two or more equations, having two or more unknown quantities, the ſame irrationality will be contained in the correſpondent values of each of the unknown quantities, unleſs two or more values of one of them are equal, &c. The ſame obſervation is alſo applied to the coefficients of an equation deduced from a given equation. 20. In the Miſcell. was publiſhed a new method of exterminating, from a given equation, irrational quantities, by finding the the multipliers, which, multiplied into it, give a rational product. 21. In the Medit. 1770, are given the different reſolutions of a certain quantity $(a^2 + rb^2)^{2m+1}$ and $(a^2 + rb^2)^{2m+2}$ into quantities of the ſame kind. 22. Mr. DE LA GRANGE has very elegantly demonſtrated Mr. WILSON's celebrated property of prime numbers contained in my book. In the laſt edition of the Medit. the ſame property is demonſtrated, and ſome ſimilar ones added. 23. In the Miſcell. is given a method of finding all the integral correſpondent values of the unknown quantities of a given ſimple equation, having two or more unknown quantities; and, in the Medit. 1770, are given methods of reducing ſimple and other algebraical equations into one, ſo that ſome unknown quantities may be exterminated; and if the unknown quantities of the reſulting equations be integral or rational, the unknown quantities exterminated may alſo be integral or rational. 24. In the Medit. are given rules for finding the different and correſpondent roots of an equation, whoſe reſolution is given. 25. Mr. de la Grange has recommended my new transformation of equations, publiſhed in the Miſcell. which perhaps is not leſs general nor elegant than any yet publiſhed; and in the Meditat. 1770 is given a method very uſeful in finding the co-efficients.

If either here, or in the preface to the Medit. Algebraicæ, I have aſcribed to myſelf any algebraical, or in the properties of curve lines any geometrical, or in the Medit. Analyt. any analytical invention, which has been before publiſhed by any other perſon, I can only plead ignorance of it, and ſhall on the very firſt conviction acknowledge it.

I muſt further add, that I have been able to carry my algebraical improvements into geometry; for from them, with ſome geometrical principles added, I have (unleſs I am deceived) deduced as many new properties of conic ſections and curve lines as have been publiſhed by any one ſince the great geometrician Apollonius.